Matrix Eigenvalues and Eigenvectors

Larry Caretto
Mechanical Engineering 501A

Seminar in Engineering Analysis

September 13, 2017

California State University Northridge

Outline

- Review last lecture
- Definition of eigenvalues and eigenvectors: **Ax** = λ**x**
- Use Det[**A** – **I**λ] = 0 to find eigenvalues
- Solve [**A** – **I**λ] **x** = 0 to get eigenvectors to within a multiplicative constant
- Number of different eigenvalues and eigenvectors

California State University Northridge 2

Review Last Lecture

- Gauss elimination for solving equations and determining rank (number of linearly independent rows or columns)
- Solution of **Ax** = **b**
  - No solutions unless rank **A** = rank [**A b**]
  - Unique if rank **A** = rank [**A b**] = number of unknowns (infinite if rank < unknowns)
  - Homogenous equations, **Ax** = **0**: only solution is **x** = **0** unless Det **A** = 0 (same as saying Rank **A** < n)

California State University Northridge 3

Uses of Eigenvalues

- In electrical and mechanical networks, provides fundamental frequencies
- Shows coordinate transformations appropriate for physical problems
- Provides way to express network problem as diagonal matrix
- Transformations based on eigenvectors used in some solutions of **Ax** = **b**

California State University Northridge 4

Eigenvalues and Eigenvectors

- Basic definition (**A** square)**: Ax** = λ**x**
- **x** is eigenvector, λ is eigenvalue
- Basic idea is that eigenvector is special vector of matrix **A**; multiplication of **x** by **A** produces **x** multiplied by a constant
- **Ax** = λ**x** => **Ax** – λ**x** = [**A** – **I**λ]**x** = **0**
- Homogenous equations; requires Det [**A** – **I**λ] = 0 for solution other than **x** = **0**

California State University Northridge 5

Det[**A** – **I**λ] = 0

$$Det[\mathbf{A}-\mathbf{I}\lambda]=Det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix}=0$$

- Det[**A** – **I**λ] = 0 produces an n^{th} order equation that has n roots for λ. May have duplicate roots for eigenvalues.

California State University Northridge 6

Two-by-two Matrix Eigenvalues

- Quadratic equation with two roots for eigenvalues $\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} =$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{21}a_{12} =$$

$$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0$$

- Eigenvalue solutions

$$\lambda = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{21}a_{12})}}{2}$$

Det **A**

Northridge 7

Two-by-two Matrix Eigenvalues

- Write $\sqrt{(a_{11}+a_{22})^2 - 4 Det\,\mathbf{A}}$ as $\sqrt{\ }$
- Add the two solutions to get

$$\lambda_1 + \lambda_2 = \frac{(a_{11}+a_{22}) + \sqrt{\ }}{2} + \frac{(a_{11}+a_{22}) - \sqrt{\ }}{2} = a_{11} + a_{22}$$

- Multiply the two solutions to get

$$\lambda_1\lambda_2 = \left(\frac{(a_{11}+a_{22}) + \sqrt{\ }}{2}\right)\left(\frac{(a_{11}+a_{22}) - \sqrt{\ }}{2}\right) = \frac{(a_{11}+a_{22})^2 - \left(\sqrt{\ }\right)^2}{4}$$

$$= \frac{(a_{11}+a_{22})^2 - (a_{11}+a_{22})^2 + 4 Det\,\mathbf{A}}{4} = Det\,\mathbf{A}$$

Northridge 8

Sum and Product

- The results on the previous slide apply to all matrix eigenvalues
- The sum of the eigenvalues is the sum of the diagonal elements of the matrix, called the trace of the matrix
- The product of the eigenvalues is the determinant of the matrix

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = Trace\,\mathbf{A} \qquad \prod_{i=1}^{n} \lambda_i = Det\,\mathbf{A}$$

Northridge 9

Two-by-two Matrix Eigenvectors

- Two eigenvectors: $\mathbf{x}_{(1)} = [x_{(1)1}\ x_{(1)2}]^T$ and $\mathbf{x}_{(2)} = [x_{(2)1}\ x_{(2)2}]^T$ ($\mathbf{x}_{(j)} = [x_{(j)1}\ x_{(j)2}]^T$)
- Substitute each eigenvalue solution, λ_j, into $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = \mathbf{0}$ to find all $\mathbf{x}_{(j)}$ components

$$(a_{11} - \lambda_j)x_{(j)1} + a_{12}x_{(j)2} = 0$$

$$a_{21}x_{(j)1} + (a_{22} - \lambda_j)x_{(j)2} = 0$$

Notation: y_i is component i of vector **y**; $\mathbf{z}_{(k)}$ is one vector in a set of vectors with components $z_{(k)i}$

Northridge 10

Two-by-two Eigenvectors II

- Eigenvector equations are homogeneous, so eigenvectors are determined only within a multiplicative constant
- Pick $x_{(j)1} = \alpha$ (arbitrary)

$$(a_{11} - \lambda_j)x_{(j)1} + a_{12}x_{(j)2} = 0$$

$$a_{21}x_{(j)1} + (a_{22} - \lambda_j)x_{(j)2} = 0$$

$$x_{(j)1} = \alpha \qquad x_{(j)2} = \frac{(\lambda_j - a_{11})}{a_{12}}\alpha = \frac{a_{21}}{(\lambda_j - a_{22})}\alpha$$

Northridge 11

Two-by-two Example

- Find eigenvalues and eigenvectors of **A** $\qquad \mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$

$$Det[\mathbf{A} - \mathbf{I}\lambda] = \begin{vmatrix} 1-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$Det[\mathbf{A} - \mathbf{I}\lambda] = (1-\lambda)(2-\lambda) - (0)(5) = 0$$

- Solutions are $\lambda_1 = 2$ and $\lambda_2 = 1$

Northridge 12

Two-by-two Example Continued

- Find $\mathbf{x}_{(1)}$ components for $\lambda_1 = 2$ $\quad \mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$
- Solve $[\mathbf{A} - \lambda\mathbf{I}]\mathbf{x} = \mathbf{0}$ for $\mathbf{x}_{(1)}$ components

$$(1-2)x_{(1)1} + 5x_{(1)2} = -x_{(1)1} + 5x_{(1)2} = 0$$
$$0x_{(1)1} + (2-2)x_{(1)2} = 0x_{(1)1} + 0x_{(1)2} = 0$$

- One equation in two unknowns
- Pick $x_{(1)2} = \alpha$ then $x_{(1)1} = 5\alpha$ from first equation
- Eigenvector $\mathbf{x}_{(1)}$ is $[5\alpha\ \alpha]^T$

Northridge 13

Two-by-two Example Concluded

- Next find $\mathbf{x}_{(2)}$ components for $\lambda_2 = 1$ $\quad \mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$
- Same as approach for finding $\mathbf{x}_{(1)}$

$$(1-1)x_{(2)1} + 5x_{(2)2} = 0x_{(2)1} + 5x_{(2)2} = 0$$
$$0x_{(2)1} + (2-1)x_{(2)2} = 0x_{(2)1} + 1x_{(2)2} = 0$$

- Both equations give $x_{(2)2} = 0$
- Pick $x_{(2)1} = \beta$ (With $x_{(2)2} = 0$, $x_{(2)1}$ can be any value and still satisfy each equation)
- $\mathbf{x}_{(2)} = [\beta\ \ 0]^T$

Northridge 14

Check Two-by-two Example

$$\mathbf{A}\mathbf{x}_{(1)} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} (1)(5\alpha)+(5)(\alpha) \\ (0)(5\alpha)+(2)(\alpha) \end{bmatrix}$$
$$= \begin{bmatrix} 10\alpha \\ 2\alpha \end{bmatrix} = 2\begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix} = \lambda_1\mathbf{x}_{(1)}$$
$$\mathbf{A}\mathbf{x}_{(2)} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \beta \\ 0 \end{bmatrix} = \begin{bmatrix} (1)(\beta)+(5)(0) \\ (0)(\beta)+(2)(0) \end{bmatrix}$$
$$= \begin{bmatrix} \beta \\ 0 \end{bmatrix} = 1\begin{bmatrix} \beta \\ 0 \end{bmatrix} = \lambda_2\mathbf{x}_{(2)}$$

Northridge 15

Eigenvector Factors

- 2 x 2 example showed $\mathbf{A}\mathbf{x}_{(j)} = \lambda_j\mathbf{x}_{(j)}$ regardless of choice of α and β
- This is general result
- We can pick one eigenvector compo-nent; typical choices are to make eigenvector simple or a unit vector (with unit length)

$$\mathbf{x}_{(1)} = \begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix} \quad \mathbf{x}_{(1)} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \mathbf{x}_{(1)} = \begin{bmatrix} 5/\sqrt{26} \\ 1/\sqrt{26} \end{bmatrix} \quad \mathbf{x}_{(2)} = \begin{bmatrix} \beta \\ 0 \end{bmatrix} \quad \mathbf{x}_{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Northridge 16

How Many Eigenvalues?

- An n x n matrix has k ≤ n distinct eigenvalues
- Algebraic multiplicity of an eigenvalue, M_λ, is the number of roots of Det$[\mathbf{A} - \mathbf{I}\lambda]$ = 0 that have the same root, λ
- Geometric multiplicity, m_λ, of eigenvalue is number of linearly independent eigenvectors for this λ

Northridge 17

Multiple Eigenvalue Example

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix} \quad \mathbf{A} - \mathbf{I}\lambda = \begin{bmatrix} 2-\lambda & 2 & -6 \\ 2 & -1-\lambda & -3 \\ -2 & -1 & 1-\lambda \end{bmatrix}$$

$$Det(\mathbf{A} - \mathbf{I}\lambda) = (2-\lambda)(-1-\lambda)(1-\lambda) + (2)(-1)(-6)$$
$$+(-2)(2)(-3) - (-2)(-1-\lambda)(-6) - (2)(2)(1-\lambda)$$
$$-(2-\lambda)(-1)(-3) = -\lambda^3 + 2\lambda^2 + \lambda - 2 + 12 + 12 + 12$$
$$+12\lambda - 4 + 4\lambda - 6 + 3\lambda = -\lambda^3 + 2\lambda^2 + 20\lambda + 24 = 0$$

Northridge 18

Multiple Eigenvalue Example II

$$Det(\mathbf{A}-\mathbf{I}\lambda)=(\lambda+2)(\lambda+2)(\lambda-6)=0$$

- Solutions are λ = 6, –2, –2
- λ = –2 has algebraic multiplicity of 2
- Find eigenvector(s) from $(\mathbf{A} - \mathbf{I}\lambda_k)\mathbf{x}_{(k)} = \mathbf{0}$

$$\begin{bmatrix} 2-\lambda_k & 2 & -6 \\ 2 & -1-\lambda_k & -3 \\ -2 & -1 & 1-\lambda_k \end{bmatrix}\begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Look at $\lambda_k = -2$

Northridge 19

Multiple Eigenvalue Example III

$$\begin{bmatrix} 2-(-2) & 2 & -6 \\ 2 & -1-(-2) & -3 \\ -2 & -1 & 1-(-2) \end{bmatrix}\begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & -6 \\ 2 & 1 & -3 \\ -2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Apply Gauss elimination to these equations

$$\begin{bmatrix} 4 & 2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Pick $x_{(k)3}$ ***and*** $x_{(k)2}$

Northridge 20

Multiple Eigenvalue Example IV

$$\begin{bmatrix} 4 & 2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Pick $x_{(k)2}$ and $x_{(k)3}$ then

$$x_{(k)1}=\frac{6x_{(k)3}-2x_{(k)2}}{4}$$

- Pick $x_{(k)3}$ = 2 and $x_{(k)2}$ = 0 => $x_{(k)1}$ = 3
- Pick $x_{(k)3}$ = 0 and $x_{(k)2}$ = 2 => $x_{(k)1}$ = -1
- Two linearly independent eigenvectors for λ = -2

$$\mathbf{x}_{(1)}=\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{x}_{(2)}=\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Northridge 21

Continue Example for λ_3 = 6

$$(\mathbf{A}-\mathbf{I}\lambda_3)\mathbf{x}_{(3)}=\begin{bmatrix} 2-6 & 2 & -6 \\ 2 & -1-6 & -3 \\ -2 & -1 & 1-6 \end{bmatrix}\begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix}=\mathbf{0}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & -6 \\ 2 & -7 & -3 \\ -2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Apply Gauss elimination to these equations

$$\begin{bmatrix} -4 & 2 & -6 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Pick $x_{(3)3}$ =1 => $x_{(3)2}$ = -1

Northridge

Example Results

$$\begin{bmatrix} -4 & 2 & -6 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_{(3)1}=\frac{6x_{(3)1}-2x_{(3)1}}{-4}=\frac{6(1)-2(-1)}{-4}=-2$$

- Eigenvalues λ_1 = -2, λ_2 = -2, λ_3 = 6 have eigenvectors shown below

$$\mathbf{x}_{(1)}=\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{x}_{(2)}=\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{x}_{(3)}=\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

Northridge 23

Eigenvector Linear Dependence

- Can we have $\alpha_1\mathbf{x}_{(1)} + \alpha_2\mathbf{x}_{(2)} + \alpha_3\mathbf{x}_{(3)} = \mathbf{0}$ without $\alpha_1 = \alpha_2 = \alpha_3 = 0$?

$$\alpha_1\mathbf{x}_{(1)}+\alpha_1\mathbf{x}_{(2)}+\alpha_3\mathbf{x}_{(3)}=\alpha_1\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}+\alpha_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}+\alpha_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}=\mathbf{0}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Homogenous equations have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ if matrix has full rank

Northridge 24

Linear Dependence II

- Matrix has full rank it its determinant is not zero

$$Det\begin{bmatrix}3 & -1 & -2\\ 0 & 2 & -1\\ 2 & 0 & 1\end{bmatrix} = \begin{matrix}(3)(2)(1)+(0)(0)(-2)+\\ (2)(-1)(-1)-(2)(2)(-2)-\\ (0)(-1)(1)-(3)(0)(-1)=15\end{matrix}$$

- Since determinant is not zero, the only solution is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so eigenvectors are linearly independent

California State University Northridge 25

In This Example **AX** = **XΛ**

$$\mathbf{AX} = \begin{bmatrix}2 & 2 & -6\\ 2 & -1 & -3\\ -2 & -1 & 1\end{bmatrix}\begin{bmatrix}3 & -1 & -2\\ 0 & 2 & -1\\ 2 & 0 & 1\end{bmatrix} = \begin{bmatrix}-6 & 2 & -12\\ 0 & -4 & -6\\ -4 & 0 & 6\end{bmatrix}$$

$$\mathbf{X}\Lambda = \begin{bmatrix}3 & -1 & -2\\ 0 & 2 & -1\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 6\end{bmatrix} = \begin{bmatrix}-6 & 2 & -12\\ 0 & -4 & -6\\ -4 & 0 & 6\end{bmatrix}$$

Λ is diagonal matrix of eigenvalues
Will show this as general result next class

California State University Northridge 26

Quadratic Forms

- Use general notation for transpose of column vector $x^T_{1m} = x_{m1}$

$$x^T = [x^T_{11} \quad x^T_{12} \quad \cdots \quad x^T_{1n}] \qquad x = \begin{bmatrix}x_{11}\\ x_{21}\\ \vdots\\ x_{n1}\end{bmatrix}$$

- Write general quadratic formula as a vector formula with symmetric a_{ik}

$$Q = \sum_{i=1}^N \sum_{k=1}^N a_{ik}\, x_i x_k = x^T A x$$

$$(Ax)_{i1} = \sum_{k=1}^N a_{ik} x_{k1} \quad x^T A x = \sum_{i=1}^N x^T_{1i} (Ax)_{i1}$$

$$x^T A x = \sum_{i=1}^N \sum_{k=1}^N x^T_{1i} a_{ik} x_{k1} = \sum_{i=1}^N \sum_{k=1}^N a_{ik} x_i x_k$$

California State University Northridge 27

Quadratic Forms II

- Theorem 2 in Chapter 8 of Kreyszig says that symmetric matrices have orthogonal eigenvectors
  - For an orthogonal matrix, $\mathbf{X}^{-1} = \mathbf{X}^T$
  - Can get *orthonormal* eigenvectors
- Theorem 4 in Chapter 8 of Kreyszig shows that if an (n x n) matrix, **A**, has a basis set of eigenvectors as columns in an (n x n) matrix, **X**, **D** = $\mathbf{X}^{-1}$**AX** is a diagonal matrix of eigenvalues

California State University Northridge 28

Quadratic Forms III

- If **D** = $\mathbf{X}^{-1}\mathbf{AX}$, then **XD** = $\mathbf{XX}^{-1}\mathbf{AX}$ = **AX** and $\mathbf{XDX}^{-1} = \mathbf{AXX}^{-1} = \mathbf{A}$
- Quadratic forms, Q = $\mathbf{x}^T\mathbf{Ax}$, will have a symmetric **A** matrix, which will have an orthonormal eigenvalue set: $\mathbf{X}^{-1} = \mathbf{X}^T$
- For Q = $\mathbf{x}^T\mathbf{Ax}$, with **A** = $\mathbf{XDX}^{-1} = \mathbf{XDX}^T$ if **X** is orthonormal, $\mathbf{X}^{-1} = \mathbf{X}^T$ so Q = $\mathbf{x}^T\mathbf{Ax}$ = $\mathbf{x}^T\mathbf{XDX}^T\mathbf{x}$
- Define **y** = $\mathbf{X}^T\mathbf{x} = \mathbf{X}^{-1}\mathbf{x}$ so **x** = **Xy**

California State University Northridge 29

Quadratic Forms IV

- From Q = $\mathbf{x}^T\mathbf{XDX}^T\mathbf{x}$ and **y** = $\mathbf{X}^T\mathbf{x}$: Q = $\mathbf{x}^T\mathbf{XDy}$
- We can write $\mathbf{x}^T\mathbf{X} = (\mathbf{X}^T\mathbf{x})^T = \mathbf{y}^T$ so Q = $\mathbf{y}^T\mathbf{Dy}$

$$Q = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

$$Q = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}\lambda_1 y_1\\ \lambda_2 y_2\\ \lambda_3 y_3\end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

y_k are principal coordinates

California State University Northridge 30

Sample Problem

- Find principal coordinates for quadratic equation: Q = $\mathbf{x}^T\mathbf{A}\mathbf{x}$ = 128, where

$$\boldsymbol{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Solution steps:
  - Find eigenvalues (λ_1, λ_2) and normalized eigenvector matrix **X** = [$\mathbf{x}_{(1)}$, $\mathbf{x}_{(2)}$] for **A**
  - Find principal coordinates, **y** = $\mathbf{X}^T\mathbf{x}$
  - Solution is Q = $\mathbf{y}^T\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2$

Northridge 31

Sample Problem II

- Find principal coordinates for quadratic equation: Q = **Ax** = 128

$$\boldsymbol{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boldsymbol{Det}(\boldsymbol{A} - I\lambda) = (17-\lambda)^2 - (15)^2 = 0 \qquad \lambda = \begin{cases} \mathbf{2} \\ \mathbf{32} \end{cases}$$

- Find eigenvector for λ_1 = 2

$$(\boldsymbol{A} - I\lambda)\boldsymbol{x} = \begin{bmatrix} 17-2 & -15 \\ -15 & 17-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Northridge ***Both equations give*** $x_1 = x_2$ 32

Sample Problem III

- Find eigenvector for λ_2 = 32

$$(\boldsymbol{A} - I\lambda)\boldsymbol{x} = \begin{bmatrix} 17-32 & -15 \\ -15 & 17-32 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Both equations give $x_1 = -x_2$

- For both eigenvectors pick x_2 = 1 to get eigenvectors $\mathbf{x}_{(1)}$ = [1 1]^T; $\mathbf{x}_{(2)}$ = [-1 1]^T
- Divide by $\sqrt{1^2+1^2} = \sqrt{2}$ for normalized $\mathbf{x}_{(1)}$ = [1/$\sqrt{2}$ 1/$\sqrt{2}$]^T; $\mathbf{x}_{(2)}$ = [-1/$\sqrt{2}$ 1/$\sqrt{2}$]^T

Northridge 33

Sample Problem IV

- Get **X** from normalized eigenvectors and find **y** = $\mathbf{X}^T\mathbf{x}$

$$\boldsymbol{X} = [x_{(1)} \quad x_{(2)}] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boldsymbol{X}^T\boldsymbol{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$$

- Q = 128 in terms of eigenvalues and y's

Q = $\mathbf{y}^T\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{2}y_1^2 + \mathbf{32}y_2^2 = \mathbf{128}$

Northridge $\mathbf{2}y_1^2/\mathbf{128} + \mathbf{32}y_2^2/\mathbf{128} = \mathbf{1}$ 34

Sample Problem V

- Result can be modified into equation for an ellipse

$$\frac{Q}{\mathbf{128}} = \frac{\mathbf{2}y_1^2}{\mathbf{128}} + \frac{\mathbf{32}y_1^2}{\mathbf{128}} = \mathbf{1}$$

$$\frac{y_1^2}{\mathbf{64}} + \frac{y_1^2}{\mathbf{4}} = \mathbf{1}$$

$$\frac{y_1^2}{\mathbf{8}^2} + \frac{y_1^2}{\mathbf{2}^2} = \mathbf{1}$$

Northridge 35